Limit quantum efficiency for violation of Clauser-Horne Inequality for qutrits

M. Genovese[*]

Istituto Elettrotecnico Nazionale Galileo Ferraris, Strada delle Cacce 91, 10135 Torino, Italy

In this paper we present the results of numerical calculations about the minimal value of detection efficiency for violating the Clauser - Horne inequality for qutrits. Our results show how the use of non-maximally entangled states largely improves this limit respect to maximally entangled ones. A stronger resistance to noise is also found.

I. INTRODUCTION

Quantum Mechanics presents various specific properties that strongly differentiate it from Classical Mechanics. In the last years these characteristic properties have been used to develop a new research field called Quantum Information [1], devoted to study codification, elaboration and transmission of information by means of quantum states with promising technological applications. One of the more relevant properties for these realizations is entanglement, namely the existence of quantum correlations that cannot be reproduced by any realistic (classical) theory based on local observables [2]. Besides the relevance for Quantum Information, these correlations have allowed various experimental tests of Standard quantum Mechanics against Local Realistic Theories [3, 4]. In particular various experiments have tested Bell inequalities [5, 6, 7] giving strong indications against local realism, even if no conclusive experiment was possible due to low detection efficiency (on the other hand locality loophole has been closed [7]).

In the last years it is emerged that the use of higher dimension Hilbert spaces ($d > 2$), qudits, instead of the traditional $d = 2$ ones, qubits, can be of interest in various applications to Foundations of Quantum Mechanics and Quantum Information.

For example, it has been shown that quantum communication based on qudits presents a higher security than the one with traditional qubit schemes [10].

Furthermore, it has also been shown that a larger violation of Bell-like inequalities is expected for qudits [12, 13, 14, 15, 16, 17, 19].

These two results have then be related in Ref. [11].

More in details, concerning Bell inequalities, various studies were addressed to understand the limit quantum efficiency for a loophole-free test of local realism (LR) and the resistance to noise.

For example, in Ref. [12] Bell inequalities with enhanced resistance to detector inefficiency were investigated. This is of particular interest since the loophole due to low detection efficiency η of the detection apparatuses is the last unsolved problem for a conclusive test of local realism [3, 4] [29]. The result was that the limit for the smallest detection efficiency η^* necessary for a loophole free test of LR, decreases for $d > 2$ maximally entangled states of a $1 - 2\%$ respect to the value $\eta^* = 82.84\%$ for $d = 2$ maximally entangled states with 2x2 number of settings of the detection apparatuses. In Ref. [13] it was then shown that for a specific hidden variable model differences between Quantum Mechanics (QM) and Local Realistic Theories (LRT) are observable up to $\eta^* > \frac{M_A + M_B - 2}{M_A M_B - 1}$, where M_A and M_B are the number of measurements available to the two experimenters (usually dubbed Alice and Bob) sharing two subsystems of a general entangled state. Finally, an asymptotic result for large d was obtained in Ref. [14].

On the other hand, the resistance to noise of some specific Bell inequalities tested by using maximally entangled states generated by multiport beam splitters was investigated in Ref. [15, 16], showing how it increases with d. In Ref. [16] it was also shown how, for maximally entangled states, the limit detection efficiency decreases from 0.8285 for $d = 2$ up to 0.8080 for $d = 16$ (being 0.8209 for qutrits, $d = 3$). In Ref. [17] a specific Clauser-Horne like inequality was proposed and investigated for the previous maximally entangled system (inequality that includes also the ones presented in [18]).

Similar results concerning the resistance to noise of LR tests performed with qudits were obtained in Ref. [19] as well.

Finally, first experimental test of Bell Inequalities with qutrits have been recently realized [20, 21, 22].

Considering the large interest both in the fields of Quantum Information [1] and of Foundations of Quantum Mechanics [3] for a characterization of violations of Local Realism in $d > 2$ Hilbert spaces, in this paper we numerically analyze the detection efficiency limit for non-maximally entangled states for the Clauser-Horne inequality proposed

[*]Electronic address: genovese@ien.it

in Ref. [17]. The interest for studying non-maximally entangled states is a consequence of the fact that for qubits it was shown [25] that a smaller detection efficiency, 66.7%, is needed for closing the detection loophole than the one required for maximally entangled states, 82.8% (for an experimental test of $d = 2$ Clauser-Horne inequality with non-maximally entangled states see Ref. [6]).

In order to give a first hint to the effect of using non-maximally entangled states, we will consider two examples: one is given by entangled systems built by multiport beam-splitters (interferometers), a second is given by an entangled state generated by using biphoton states [23]. Both these qutrits states have already been realized with photons [22, 24] (for the first also entanglement has been obtained [22]) and, therefore, our results, showing a relevant improvement by using non-maximally entangled states, are of a large relevance for future experimental tests of LRT against QM and applications to quantum information.

II.   CLAUSER-HORNE INEQUALITY FOR QUTRITS

The Clauser-Horne inequality, valid for LRT, introduced in Ref. [17] is

$$
\begin{aligned}
CH = P^{11}(2,1) + P^{12}(2,1) - P^{21}(2,1) + P^{22}(2,1)+ \\
P^{11}(1,2) + P^{12}(1,2) - P^{21}(1,2) + P^{22}(1,2)+ \\
P^{11}(2,2) + P^{12}(1,1) - P^{21}(2,2) + P^{22}(2,1)- \\
\left(P_1^1(1) + P_1^1(2) + P_2^2(1) + P_2^2(2)\right) <= 0
\end{aligned}
\tag{1}
$$

where $P^{ij}(k,l)$ denotes the joint probability for Alice measuring in the basis i and Bob in the basis j $(i,j = 1,2)$ a thricotomic observable to obtain the result k and l respectively $(k,l = 1,2,3)$, whilst $P_n^i(k)$ denotes the single measurement probability of Alice $(n = 1)$ or Bob $(n = 2)$ to obtain the result k when the basis i is used [30].

While $CH <= 0$ for every LRT, in Ref. [17] was shown that this inequality is violated (with a maximal violation $CH = 0.29098$) by a qutrit maximally entangled state.

This state is obtained by applying a tritter (unbiased 6-port beamsplitter) plus three phase shifts, which altogether are described by the unitary transformation

$$
U_{kl} = \frac{1}{\sqrt{3}} exp(i2\pi/3(k-1)(l-1)) \cdot exp(i\phi_l),
\tag{2}
$$

to the state

$$
\Psi = \frac{1}{\sqrt{3}}(|1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B + |3\rangle_A|3\rangle_B)
\tag{3}
$$

where the states $|i\rangle_{A,B}$ describe the i-th basis state of the qutrit, respectively sent to Alice and Bob.

The measurement is described by the projection in one of these states $|i\rangle$. Thus the quantum probabilities are:

$$
P^{ij}(k,l) = |\langle k_A|\langle l_B|U_A(\vec{\phi_i})U_B(\vec{\theta_j})|\Psi\rangle|^2,
\tag{4}
$$

$$
P_n^i(k) = |\langle k_n|U_n(\vec{\phi_i})|\Psi\rangle|^2
\tag{5}
$$

The result of a violation of the Clauser-Horne inequality, $CH = 0.29098$, corresponds to a limit for the minimal detection efficiency of $\eta^* = 0.8209$ in perfect agreement with Ref. [16].

In the following we will reconsider this violation for non-maximally entangled states with the purpose of verifying if also in this case, as in the qubits one [25], a substantial improvement on the value of the minimal detection efficiency appears.

III.   NUMERICAL RESULTS WITH NON-MAXIMALLY ENTANGLED STATES

Let us begin by reconsidering the violation of inequality 1 in the case of a non-maximally entangled state obtained by the transformation 2 applied to the state (a,b are chosen to be real):
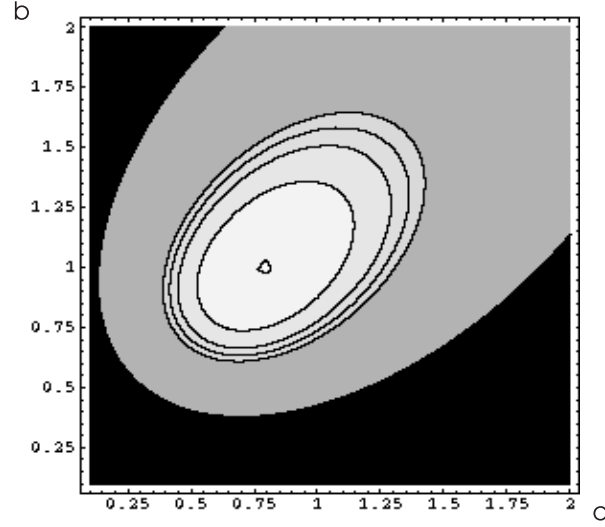
FIG. 1:
Contour plot of the quantity CH (see Eq. 1), calculated with the qutrit state generated by the tritter scheme, in the plane a (abscissa) - b (ordinate) in a region around a maximum for a detection efficiency $\eta = 0.85$. Contour lines are at -0.1,0,0.007,0.015,0.03,0.05.

$$\Psi = \frac{1}{\sqrt{1 + a^2 + b^2}} \left(|1\rangle_A |1\rangle_B + a|2\rangle_A |2\rangle_B + b|3\rangle_A |3\rangle_B \right) \tag{6}$$

This state could be easily obtained with a simple (unbalanced tritter) modification of the experimental scheme of Ref. [22].

By using analytic expressions for coincidence probabilities deriving by 6 we perform a numerical maximization (by using the software Mathematica) on the phases $\vec{\phi}$ and $\vec{\theta}$ and on the parameters a, b.

Our numerical simulation shows that the limit for the detection efficiency $\eta^* = 0.8209$ for maximally entangled states, can be lowered to $\eta^* = 0.8139$ for non-maximal entanglement. This result is not particularly large, even if it is analogous to the ones obtained increasing the dimension d of the Hilbert space and/or the number of measurements reported in Ref. [12, 16].

In order to give an idea of the region where the inequality 1 is violated, in Fig.1 we plot the value of the quantity CH, Eq. 1, around one maximum in function of the two parameters a, b for a detection efficiency $\eta = 0.85$.

In Fig.2 the same contour plot is shown for the case $\eta = 0.82$. It is evident that the point $a = 1, b = 1$ (maximal entanglement), which is eccentric from the maximum but still in a positive region for $\eta = 0.85$, becomes just outside the positive region for $\eta = 0.82$.

The relatively small numerical relevance of using non-maximal entangled states can be due to the specific choice of the entangled state. In particular, in this case the single detector probabilities $P_n^i(k)$ are independent on the parameters a and b and equal to $1/3$. Thus the last line of inequality 1 is constant, giving a relatively large negative contribution $-4/3$ (we will see later than in other cases the contribution of this part can be reduced). The use of other states can change this situation and eventually improve the result about the limit detection efficiency.

Let us analyze more in details this point, beginning with the qubit case. For the $d = 2$ non-maximally entangled state:

$$|\Psi\rangle = \frac{|0\rangle|0\rangle + a|1\rangle|1\rangle}{\sqrt{1 + |a|^2}} \tag{7}$$

the original Clauser-Horne sum is

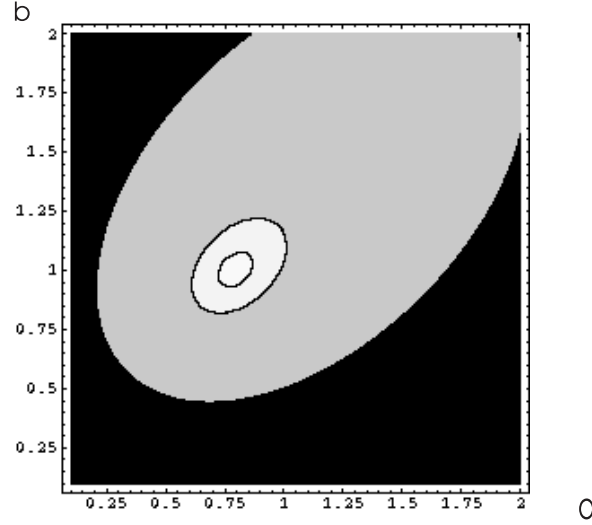$$CH(d = 2) = p(\theta_1, \theta_2) - p(\theta_1, \theta_2') +$$

FIG. 2: Contour plot of the quantity CH (see Eq. 1), calculated with the qutrit state generated by the tritter scheme, in the plane $a - b$ in a region around a maximum for a detection efficiency $\eta = 0.82$. Contour lines are at -0.1,0,0.007.

$$p(\theta'_1, \theta_2) + p(\theta'_1, \theta'_2) - p(\theta'_1) - p(\theta_2) \tag{8}$$

which is strictly negative for every local realistic theory. In (8), $p(\theta_1, \theta_2)$ is the probability of coincidences between channels 1 and 2 when θ_1 and θ_2 selections are performed, $p(\theta_i)$ are single detector count probabilities corresponding to the selection θ_i. In Fig. 3 we report the contour plot for CH in the qubit case in the plane $\eta - a$

If one considers a maximally entangled state ($a = 1$) for $\eta = 1$ the inequality is violated by a quantity $CH = 0.2071$. The (negative) contribution from the single particle probabilities, CH_{single} is 82.85% of the joint probabilities one, CH_{joint}.

When η is decreased, for example to 0.85, for the maximally entangled state the violation is reduced to $CH = 0.0221$ and now the contribution CH_{single} is the 97.4% of CH_{joint}. If the entanglement parameter a is varied as well the maximal violation is $CH(\eta = 0.85) = 0.0496$ for $a = 0.608$. In this case the contribution of CH_{single} respect to CH_{joint} is reduced to 90.76%.

On the other hand for the qutrit case just considered, the ratio CH_{single}/CH_{joint} is 82.1% for a maximally entangled state when $\eta = 1$ and becomes $CH_{single}/CH_{joint} = 0.9657$ for $\eta = 0.85$ ($CH = 0.0402$). In this second case, for a non-maximally entangled state it only reduces to $CH_{single}/CH_{joint} = 0.9575$ ($CH = 0.05033$).

In order to better investigate this point, we have therefore considered as a second example the case where qutrit basis is realized by using degenerate biphotons [23, 24]

$$|1\rangle = |HH\rangle$$
$$|2\rangle = |HV\rangle$$
$$|3\rangle = |VV\rangle \tag{9}$$

where H and V denote horizontal and vertical polarization of photons respectively. In this specific case the two observable pertains to two independent photons and therefore the detection efficiency for the single state, here a biphoton, corresponds to the product of the quantum efficiencies of two photon-detectors. Nevertheless, since we are interested in a general theoretical discussion about how large can be the effect of using non-maximally entangled states for qutrits case (in a general abstract Hilbert space), in the following we discuss the results in term of the detection efficiency for the state (of course, the square root of it immediately gives the single photon-detector one).

The measurement, performed on the non-maximally entangled state 6, is both for Alice and Bob, represented by a polarization selection on a chosen basis at a certain angle θ from the horizontal direction. The results of this measurement will be:

i) both the photons passing the selection (e.g. both up if a polarizing beam splitter is used),
ii) both not passing it (e.g. both down)
iii) one passing and the other not the selection (e.g. one up and one down)
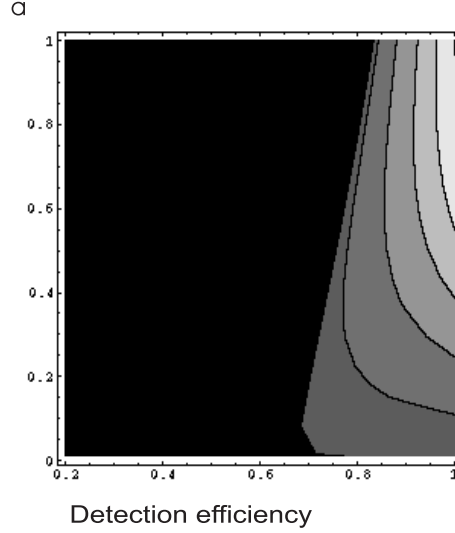
a

Detection efficiency

FIG. 3: Contour plot of the quantity $CH(d=2)$ (see Eq. 8) in the plane with a (non maximally entanglement parameter, see the text for the definition) as ordinate and η (total detection efficiency) as abscissa. The leftmost region (in black) corresponds to the region where no detection loophole free test of Bell inequalities can be performed. The contour lines are at $0, 0.01, 0.05, 0.1, 0.15, 0.2$. One can observe how the lowest quantum efficiency for having $CH > 0$ is 0.667 for $a \approx 0.15$.

These cases are respectively represented by the projectors:

$$
\begin{aligned}
P_1 &= |2_\theta\rangle\langle 2_\theta| \\
P_2 &= |2_{\theta+\pi/2}\rangle\langle 2_{\theta+\pi/2}| \\
P_3 &= |1_\theta 1_{\theta+\pi/2}\rangle\langle 1_\theta 1_{\theta+\pi/2}|
\end{aligned}
\tag{10}
$$

where $|n_\theta\rangle$ denotes the state of n photons with polarization along an axis forming an angle θ with the horizontal direction.

Experimentally this state could be generated with the techniques presented in Ref. [26] for entangling in polarization four (or more) photons states.

In our numerical simulation we have initially chosen as the measurement results 1,2 in the inequality 1 the first two former cases (P_1, P_2). We have then maximized the quantity CH on the Alice and Bob bases $(\theta_A^1, \theta_A^2, \theta_B^1, \theta_B^2)$ and on the entanglement parameters a, b.

Our numerical results show that for a maximally entangled state $(a = b = 1)$ the maximal violation is $CH = 0.1765$ corresponding to a limit of the detection efficiency $\eta^* = 0.8835$. On the other hand, when a generic non maximal entangled state is chosen, the result strongly improves reaching as limit for the detection efficiency $\eta^* = 0.76$.

For the sake of exemplification, in Fig.4 we plot the value of the quantity CH, Eq. 1, in function of the two parameters a, b for a detection efficiency $\eta = 0.85$.

The effect of using non-maximally entangled states is therefore rather large, even if still smaller than the effect for qubits (where it is lowered [25] from $\eta^* = 0.828$ to $\eta^* = 0.667$). In particular, the effect is larger here than for qutrits generated by using tritters.

This last result is related to the fact that in this second case the contribution from the single probabilities decreases relevantly for a non-maximally entangled state. Our results are that for a maximally entangled state this contribution increases from 88.31% when $\eta = 1$ to 98.12% when $\eta = 0.9$ ($CH = 0.02297$). On the other hand, leaving a, b free to vary we obtain at $\eta = 1$ a maximal violation $CH = 0.1851$ with $CH_{single}/CH_{joint} = 0.867$. At $\eta = 0.9$ single probabilities contribution becomes 91.88% of the joint probabilities one (with $CH = 0.06166$ when $a = 1.820, b = 1.002$ [31]).

Even a larger effect is obtained when the case P_1, P_3 is chosen. For the maximally entangled state $a = b = -1$ we find $CH = 0.21007$ for $\eta = 1$, with a ratio $CH_{single}/CH_{joint} = 0.8507$. On the other hand when the parameters a, b are left free to vary, we obtain a much larger violation, $CH = 0.31607$, corresponding to a reduction of the ratio CH_{single}/CH_{joint} to 0.7610 ($a = -1.37, b = -0.607$). For $\eta = 0.9$ we have $CH = 0.06256$ with $CH_{single}/CH_{joint} =$
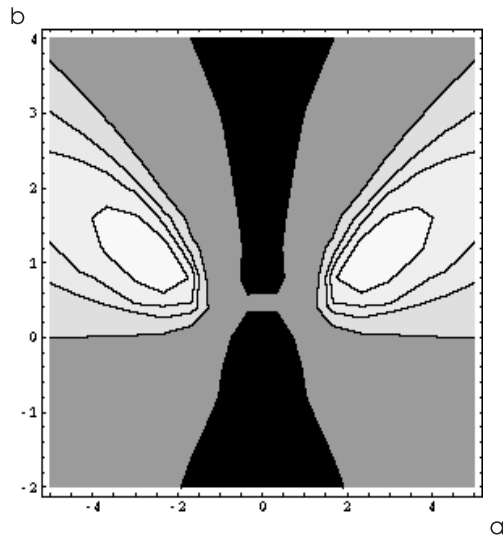
FIG. 4: Contour plot of the quantity CH (see Eq. 1), calculated for biphoton qutrits, in the plane a (abscissa) - b (ordinate) for a detection efficiency $\eta = 0.85$. Contour lines are at CH=-0.1,0,0.01,0.015,0.02.

0.945 for the maximally entangled state, whilst $CH = 0.1686$, $CH_{single}/CH_{joint} = 0.8334$ for a non-maximally entangled one. Finally, the limit for violating the Clauser-Horne inequality is $\eta^* = 0.8505$ for maximal entanglement, reduced to $\eta^* = 0.7413$ for $a = -1.755, b = -0.572$. For the sake of comparison, the best results for qubits, tritter and biphoton qutrits are shown in table 1.

In the last case also the resistance to noise is increased respect to the result of Ref. [17] for maximally entangled states, since the value of CH is increased. If, in analogy to Ref. [17], one considers the mixed state ($0 \leq F \leq 1$)

$$\rho = (1 - F)|\Psi\rangle\langle\Psi| + F\rho_{noise} \tag{11}$$

where ρ_{noise} is a diagonal matrix with entries equal to 1/9 the threshold value of F for violating the Clauser-Horne inequality is $F_{th} = 0.3216$ to be compared with $F_{th} = 0.30385$ obtained in Ref. [17, 18, 27]. Incidentally, the result that noise threshold is higher for non-maximally entangled states follows also from the results of Ref. [28], from which one obtains $F_{th} = 0.31471$.

These results further show the large effect on the violation of Clauser - Horne inequality obtained by using non-maximally entangled states. Since the maximum has been evaluated by a numerical maximization even larger effects cannot be completely excluded.

	Maximally entangled states	Non-maximally entangled states
qubits	0.828	0.667
tritter qutrits	0.8209	0.8139
biphoton qutrits	0.8505	0.7413

Table 1: limit on the detection efficiency for a certain state necessary for obtaining a detection loophole free experiment.

IV. CONCLUSIONS

In conclusion in this paper we have presented the results of numerical simulations about the minimal value of detection efficiency for violating the Clauser - Horne inequality for qutrits proposed in Ref. [17]. In particular we have considered the case of entangled qutrits generated by means of a tritter scheme (as in Ref. [17]) and of qutrits based on biphotons (as suggested in [23, 24]).

Our results show that, in analogy to the case of qubits [25], the use of non-maximally entangled states may largely improve this limit. Nevertheless, in these specific numerical examples, the effect remains smaller than in qubit case. The difference among the cases considered here show how the specific numerical results largely depend on the choice of states and measurements. Altogether, our results give a first indication of the interest of using non-maximally

entangled states also for $d > 2$ Hilbert spaces, pointing out the relevance of further theoretical and experimental studies in this sense.

[1] See for example "Quantum Computation and Quantum Information", M. A. Nielsen and I.L. Chuang, ed. Cambridge Univ. Press, Cambridge (2000). "The Physics of Quantum Information", D. Bouwmeester et al., ed. Springer Verlag, Berlin (2000).
[2] J. S. Bell, Physics **1** (1964) 195.
[3] See for example G. Auletta, *Foundations and interpretation of quantum mechanics* (World Scientific, Singapore 2000) and Ref.s therein.
[4] M. Genovese in "Progress in Experimental High Energy Physics Research" , (Nova Science, 2004) in press and Ref.s therein.
[5] A. Aspect et al., Phys. Rev. Lett. 49 (1982) 1804; J. P. Franson, Phys. Rev. Lett. 62 (1989) 2205; J. G. Rarity and P. R. Tapster, Phys. Rev. Lett. 64 (1990) 2495; J. Brendel et al., Eur.Phys.Lett. 20 (1992) 275; P. G. Kwiat et al., Phys. Rev. A 41 (1990) 2910;T.E. Kiess et al., Phys. Rev. Lett. 71 (1993) 3893; P.G. Kwiat et al., Phys. Rev. Lett. 75 (1995) 4337; Z.J. Ou and L. Mandel, Phys. Rev. Lett. 61 (1988) 50; Y.H.Shih et al., Phys. Rev. A 47 (1993) 1288; J.R.Torgerson et al., Phys. Lett. A 204 (1995) 323; J.W. Pan et al., Nature 403 (2000) 515.
[6] G. Brida, M. Genovese, C. Novero and E. Predazzi, Phys. Lett. A **268**, 12, 2000; G. Brida et al., Phys. Lett. A **299**, 121, 2002;
[7] W. Tittel et al., Phys. Rev. Lett. 81 (1998) 3563; G. Weihs et al., Phys. Rev. Lett. 81 (1998) 5039.
[8] M. A. Rowe et al., Nature **409**, 791, 2001.
[9] M. Genovese, preprint IEN-12-04 in press.
[10] H. Bechmann-Pasquinucci and W. Tittel, Phys. Rev. A **61**, 062308, 2000; H. Bechmann-Pasquinucci and A. Peres, Phys. Rev. Lett. **85** , 3313, 2000; M.Bourennane et al., Phys. Rev. A **63**, 062303, 2001; N.J. Cerf et al., Phys. Rev. Lett. **88**, 127902, 2002; D. Bruss and C. Macchiavello, Phys. Rev. Lett. **88**, 127901, 2002; M. Genovese and C. Novero, *Eur. Journ. of Phys. D.* **21**, 109, 2002.
[11] A.Acin et al., quant-ph 0310166.
[12] S. Massar et al., Phys. Rev. A **66** (02) 052112.
[13] S. Massar and S. Pironio, Phys. Rev. A **68** (04) 062109.
[14] S. Massar, Phys. Rev. A **65** (02) 032121.
[15] D. Kaszlikowski et al., Phys. Rev. Lett. **85** (00) 4418;
[16] T. Durt et al., quant-ph 0101084.
[17] D. Kaszlikowski et al.,Phys. Rev. A **65** (02) 0321118.
[18] D. Collins et al., Phys. Rev. Lett. **88** (02) 040404.
[19] H.Bechmann-Pasquinucci and N. Gisin, quant-ph 0204122. D. Collins et al., Phys. Rev. Lett. **88** (02) 040404. J. C.
[20] A. Vaziri et al., Phys. Rev. Lett. 89 (02) 240401.
[21] Howell, A. Lamas-Linares, and D. Bouwmeester Phys. Rev. Lett. 88, 030401 (2002)
[22] R.T.Thew et al., quant-ph 0402048.
[23] A.V. Burlakov and D. N. Klyshko, JETP Lett. **69** (99) 795.
[24] G.A. Maslennikov et al., J. Opt. B **5** (03) S530. A.V. Burlakov et al., quant-ph 9907099. M.V. Chekhova et al., quant-ph/0311005.
[25] P. H. Eberhard, Phys. Rev. A **47**, R747, 1993.
[26] H. Weinfurter and M. Zukowski, Phys. Rev. A **64** (01) 010102; Z. Zhao et al., Phys. Rev. Lett. **91** (03) 180401, quant-ph/0402096.
[27] J. Chen et al., quant-ph 0103099.
[28] A. Acin et al., Phys. Rev. A 65 (2002) 052325; quant-ph 0111143.
[29] Incidentally, it must be noticed that a recent experiment [8] based on the use of Be ions has reached very high detection efficiencies (around 98 %), largely sufficient for closing detection loophole, but in this case not only space like separation

required for closing locality loophole was not satisfied, but the two subsystems (the two ions) were even not really separated during the measurement. Therefore, this experiment cannot be considered a real implementation of a loophole free test of Bell inequalities, even if it represents a relevant progress in this sense (see [9] for a general review on these problems).

[30] It is interesting to notice that this inequality even if valid for qutrits does not involve the third outcome of the measurement [17].

[31] Other minima correspond to other parameters values.